

On classification of dynamical r-matrices

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February 9, 2008

Abstract

Using the gauge transformations of the Classical Dynamical Yang-Baxter Equation introduced by P. Etingof and A. Varchenko in [EV], we reduce the classification of dynamical r-matrices r on a commutative subalgebra \mathfrak{l} of a Lie algebra \mathfrak{g} to a purely algebraic problem, under some assumption on the symmetric part of r . We then describe, for a simple complex Lie algebra \mathfrak{g} , all non skew-symmetric dynamical r-matrices on a commutative subalgebra $\mathfrak{l} \subset \mathfrak{g}$ which contains a regular semisimple element. This interpolates results of P. Etingof and A. Varchenko ([EV], when \mathfrak{l} is a Cartan subalgebra) and results of A. Belavin and V. Drinfeld for constant r-matrices ([BD]). This classification is similar, and in some sense simpler than the Belavin-Drinfeld classification.

1 The Classical Yang-Baxter Equation

Let \mathfrak{g} be a Lie algebra. The CYBE is the following algebraic equation for an element $r \in \mathfrak{g} \otimes \mathfrak{g}$:

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0. \quad (1)$$

Solutions of this equation are called r-matrices. In the theory of quantum groups, one is mainly interested in r-matrices satisfying

$$r + r^{21} \in (S^2 \mathfrak{g})^{\mathfrak{g}}. \quad (2)$$

See [CP] for the links with the theory of quantum groups, and [Che] for links with Conformal Field Theory and the Wess-Zumino-Witten model on \mathbb{P}^1 . The geometric interpretation of the CYBE was given by Drinfeld in terms of Poisson-Lie groups ([Dr1]).

2 The Belavin-Drinfeld Classification

Notations: Let \mathfrak{g} be a simple complex Lie algebra with a nondegenerate invariant form (\cdot, \cdot) , $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra and Δ the root system. For $\alpha \in \Delta$, let \mathfrak{g}_α denote the root subspace associated to α . Let W be the Weyl group and

s_α , $\alpha \in \Delta$ the reflection with respect to α^\perp . Finally, let $\Omega \in S^2 \mathfrak{g}$ and $\Omega_{\mathfrak{h}} \in S^2 \mathfrak{h}$ be the inverse elements to the form $(,)$. Notice that $(S^2 \mathfrak{g})^{\mathfrak{g}} = \mathbb{C}\Omega$.

For any polarization $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, we denote by Π or $\Pi(\mathfrak{n}_+)$ the corresponding set of simple positive roots, by Δ_+ the set of positive roots and by $\mathfrak{b}_\pm = \mathfrak{n}_\pm \oplus \mathfrak{h}$ the Borel subalgebras. For $\Gamma \subset \Pi$, set $\langle \Gamma \rangle = \mathbb{Z}\Gamma \cap \Delta$, and let \mathfrak{g}_Γ be the subalgebra generated by \mathfrak{g}_α , $\alpha \in \langle \Gamma \rangle$. We will write $\mathfrak{g}_\Gamma = \mathfrak{n}_+(\Gamma) \oplus \mathfrak{h}(\Gamma) \oplus \mathfrak{n}_-(\Gamma)$ for the induced polarization and $W(\Gamma)$ for the subgroup of W generated by s_α , $\alpha \in \Gamma$.

Let us fix a polarization of \mathfrak{g} .

Definition: A *Belavin-Drinfeld triple* is a triple $(\Gamma_1, \Gamma_2, \tau)$ where $\Gamma_1, \Gamma_2 \subset \Pi$ and $\tau : \Gamma_1 \xrightarrow{\sim} \Gamma_2$ is a norm-preserving bijection satisfying the following "nilpotency" condition:

"For any $\gamma_1 \in \Gamma_1$, there exists $n > 0$ such that $\tau^n(\gamma_1) \in \Gamma_2 \setminus \Gamma_1$ ".

Let $(\Gamma_1, \Gamma_2, \tau)$ be a Belavin-Drinfeld triple. For each choice of Chevalley generators $(e_\alpha, f_\alpha, h_\alpha)_{\alpha \in \Gamma_i}$, $i = 1, 2$, the isomorphism τ induces a Lie algebra isomorphism $\mathfrak{g}_{\Gamma_1} \xrightarrow{\sim} \mathfrak{g}_{\Gamma_2}$ (by $e_\alpha \mapsto e_{\tau(\alpha)}$, $f_\alpha \mapsto f_{\tau(\alpha)}$, $h_\alpha \mapsto h_{\tau(\alpha)}$). Define a partial order on Δ_+ by setting $\alpha < \beta$ if there exists $n > 0$ such that $\tau^n(\alpha) = \beta$ (in particular, $\alpha \in \Gamma_1$ and $\beta \in \Gamma_2$).

Definition: A basis $(x_\alpha)_{\alpha \in \Delta}$ of $\mathfrak{n}_+ \oplus \mathfrak{n}_-$ is called *admissible* if $(x_\alpha, x_{-\alpha}) = 1$ and $\tau(x_\alpha) = x_{\tau(\alpha)}$ for $\alpha \in \langle \Gamma_1 \rangle$.

Theorem 1 (Belavin-Drinfeld) *Let \mathfrak{g} be a simple complex Lie algebra.*

1. *Let $(\Gamma_1, \Gamma_2, \tau)$ be a Belavin-Drinfeld triple, (x_α) an admissible basis, and let $r_0 \in \mathfrak{h} \otimes \mathfrak{h}$ be such that*

$$r_0 + r_0^{21} = \Omega_{\mathfrak{h}}, \quad (3)$$

$$(\tau(\alpha) \otimes 1)r + (1 \otimes \alpha)r = 0 \quad \text{for } \alpha \in \Gamma_1. \quad (4)$$

Then

$$r = r_0 + \sum_{\alpha \in \Delta_+} x_{-\alpha} \otimes x_\alpha + \sum_{\alpha, \beta \in \Delta_+, \alpha < \beta} x_{-\alpha} \wedge x_\beta \quad (5)$$

is an r -matrix satisfying $r + r^{21} = \Omega$.

2. *Any r -matrix satisfying $r + r^{21} = \Omega$ is of the above type for a suitable polarization of \mathfrak{g} .*

This theorem is proved in [BD]. For instance, the standard r -matrix for a fixed polarization $r = \frac{\Omega_{\mathfrak{h}}}{2} + \sum_{\alpha \in \Delta_+} x_{-\alpha} \otimes x_\alpha$ corresponds to $\Gamma_1 = \Gamma_2 = \emptyset$.

Remark: Skew-symmetric r -matrices admit a well known interpretation in terms of nondegenerate 2-cocycles on Lie subalgebras of \mathfrak{g} ([Dr1]), but their classification is unavailable since it requires a classification of Lie subalgebras in \mathfrak{g} .

3 The Dynamical Yang-Baxter Equation

Let \mathfrak{g} be a Lie algebra over \mathbb{C} and $\mathfrak{l} \subset \mathfrak{g}$ a subalgebra. An element $x \in \mathfrak{g} \otimes \mathfrak{g}$ will be called \mathfrak{l} -invariant if

$$[k \otimes 1 + 1 \otimes k, x] = 0 \quad (\forall k \in \mathfrak{l}). \quad (6)$$

For $x \in \mathfrak{g}^{\otimes 3}$, we let $\text{Alt}(x) = x^{123} + x^{231} + x^{312}$. Let $D \subset \mathfrak{l}^*$ be any open region.

The CDYBE is the following differential equation for a holomorphic \mathfrak{l} -invariant function $r : D \rightarrow \mathfrak{g} \otimes \mathfrak{g}$:

$$\text{Alt}(dr) + [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0, \quad (7)$$

where the differential of r is considered as a holomorphic function

$$dr : D \rightarrow \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}, \quad \lambda \mapsto \sum_i x_i \otimes \frac{\partial r^{23}}{\partial x_i}(\lambda), \quad (\lambda \in \mathfrak{l}^*),$$

for any basis (x_i) of \mathfrak{l} . In this case,

$$\text{Alt}(dr) = \sum_i x_i^{(1)} \frac{\partial r^{23}}{\partial x_i} + \sum_i x_i^{(2)} \frac{\partial r^{31}}{\partial x_i} + \sum_i x_i^{(3)} \frac{\partial r^{12}}{\partial x_i}.$$

The solutions to this equation are called *dynamical* r -matrices. Dynamical r -matrices which are relevant to the theory of quantum groups are those satisfying the following condition, analogous to (2):

$$\text{Generalized unitarity} : r(\lambda) + r^{21}(\lambda) \in (S^2 \mathfrak{g})^{\mathfrak{g}}. \quad (8)$$

Remark: the CDYBE was first written down by G. Felder and C. Wieczorkowski in connection with the Wess-Zumino-Witten model on elliptic curves ([FW]). The relation with elliptic quantum groups is explained in [Fe]. A geometric interpretation of the CDYBE analogous to the theory of Poisson-Lie groups for the CYBE is given in [EV].

4 Gauge transformations:

We recall some results from [EV]. We suppose here that \mathfrak{l} is commutative and we let D be the formal polydisc centered at the origin. Let G be a complex Lie group such that $\text{Lie}(G) = \mathfrak{g}$, and let L be the connected subgroup of G such that $\text{Lie}(L) = \mathfrak{l}$. Let G^L be the centralizer of L in G and $\mathfrak{g}^{\mathfrak{l}}$ its Lie algebra. We will denote by $(\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{l}}$ the space of all \mathfrak{l} -invariant elements in $\mathfrak{g} \otimes \mathfrak{g}$.

Let $g : D \rightarrow G^L$ be any holomorphic function; the 1-form $\eta = g^{-1}dg$ gives rise to a function $\bar{\eta} : D \rightarrow \mathfrak{l} \otimes \mathfrak{g}^{\mathfrak{l}}$. If $r : D \rightarrow (\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{l}}$ is an \mathfrak{l} -invariant function satisfying (8), we set

$$r^g = (g \otimes g)(r - \bar{\eta} + \bar{\eta}^{21})(g^{-1} \otimes g^{-1}).$$

Proposition 1 *The function r is a dynamical r -matrix if and only if the function r^g is.*

Thus the group $\text{Map}(D, G^L)$ is a gauge transformation group for the CDYBE. Notice that this group is not commutative if G^L isn't.

Theorem 2 *Let $\rho, r : D \rightarrow \mathfrak{g}^{\otimes 2}$ be two dynamical r -matrices satisfying (8) such that $r(0) = \rho(0)$. Then there exists $g \in \text{Map}(D, G^L)$ such that $\rho = r^g$.*

This shows that the space of dynamical r -matrices is, up to gauge equivalence, finite dimensional. Proofs of the above results can be found in [EV].

We will now prove a converse of Theorem 2 which reduces the CDYBE to a purely algebraic equation under some assumption on the symmetric part $\frac{\Omega}{2}$ of r : let $\Omega \in (S^2 \mathfrak{g})^{\mathfrak{g}}$, let \mathfrak{g}_Ω be the ideal in \mathfrak{g} generated by the components of Ω and denote by $\mathfrak{g}_\Omega = \bigoplus_\lambda \mathfrak{g}_\Omega(\lambda)$ the generalized weight space decomposition of \mathfrak{g}_Ω with respect to the adjoint action of \mathfrak{l} . The condition we will need is the following:

$$\mathfrak{g}^{\mathfrak{l}} \text{ acts semisimply on } \mathfrak{g}_\Omega(0) \quad (*)$$

Suppose that $(*)$ is fulfilled and let $z(\mathfrak{g}^{\mathfrak{l}})$ denote the center of $\mathfrak{g}^{\mathfrak{l}}$. Then we have a decomposition $\mathfrak{g}_\Omega(0) = z_0(\mathfrak{g}^{\mathfrak{l}}) \oplus V$ where $z_0(\mathfrak{g}^{\mathfrak{l}}) = z(\mathfrak{g}^{\mathfrak{l}}) \cap \mathfrak{g}_\Omega(0)$ and V is the sum of all non-trivial irreducible $\mathfrak{g}^{\mathfrak{l}}$ -modules in $\mathfrak{g}_\Omega(0)$. It is clear that $\mathfrak{l} \cap V = \{0\}$. We will say that a complement \mathfrak{l}' of \mathfrak{l} in \mathfrak{g} is admissible if $V \subset \mathfrak{l}'$, and write $\pi : \mathfrak{g} \rightarrow \mathfrak{l}$ for the projection along \mathfrak{l}' . Notice that by $\mathfrak{g}^{\mathfrak{l}}$ -invariance of Ω ,

$$\Omega \in S^2 z_0(\mathfrak{g}^{\mathfrak{l}}) \oplus S^2 V \oplus \bigoplus_{\lambda \neq 0} \mathfrak{g}_\Omega(\lambda) \otimes \mathfrak{g}_\Omega(-\lambda). \quad (9)$$

We will denote by $CYB : \mathfrak{g}^{\otimes 2} \rightarrow \mathfrak{g}^{\otimes 3}$ the map:

$$r \mapsto [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}].$$

It is more convenient to work with the skew-symmetric part of r . If $r(\lambda) + r^{21}(\lambda) = \Omega \in (S^2(\mathfrak{g}))^{\mathfrak{g}}$, we set $s(\lambda) = r(\lambda) - \frac{\Omega}{2}$. It is easy to see that the CDYBE for r is equivalent to the following equation for s :

$$\text{Alt}(ds) + CYB(s) + \frac{1}{4}CYB(\Omega) = 0. \quad (10)$$

Recall that as Ω is symmetric and invariant, $CYB(\Omega) = [\Omega_{13}, \Omega_{23}]$.

Theorem 3 *Let G be a complex Lie group and $L \subset G$ a connected commutative subgroup. Let $\mathfrak{g}, \mathfrak{l}, \mathfrak{g}^{\mathfrak{l}}$ denote the Lie algebras of G, L and G^L . Let $\Omega \in (S^2 \mathfrak{g})^{\mathfrak{g}}$. Then*

1. *Let \mathfrak{l}' be any complement of \mathfrak{l} in \mathfrak{g} . Any dynamical r -matrix $r(\lambda)$ on \mathfrak{l} such that $r(\lambda) + r^{21}(\lambda) = \Omega$ is gauge equivalent to a dynamical r -matrix $\tilde{r}(\lambda) : D \rightarrow \frac{\Omega}{2} + (\Lambda^2(\mathfrak{l}'))^{\mathfrak{l}}$.*

2. Suppose that condition (*) is true and let \mathfrak{l}' be any admissible complement of \mathfrak{l} in \mathfrak{g} . Let $r_0 \in \frac{\Omega}{2} + (\Lambda^2(\mathfrak{l}'))^{\mathfrak{l}}$ satisfy

$$CYB(r_0) \in \text{Alt}(\mathfrak{l} \otimes \mathfrak{g} \otimes \mathfrak{g}) \quad (11)$$

such that $s_0 = r_0 - \frac{\Omega}{2}$ is a regular point of the algebraic manifold

$$M_{\Omega} = \{s \in (\Lambda^2(\mathfrak{l}'))^{\mathfrak{l}} \mid CYB(s + \frac{\Omega}{2}) \in \text{Alt}(\mathfrak{l} \otimes \mathfrak{g} \otimes \mathfrak{g})\}.$$

Then there exists a dynamical r -matrix $r(\lambda) : D \rightarrow \frac{\Omega}{2} + (\Lambda^2(\mathfrak{l}'))^{\mathfrak{l}}$ such that $r(0) = r_0$.

The condition (*) is satisfied in the following two interesting special cases: when $\Omega = 0$ (triangular case) or when $\mathfrak{g}^{\mathfrak{l}}$ acts semisimply on \mathfrak{g} (for instance, G is reductive and L is contained in a maximal torus of G or more generally, if G^L is reductive).

The proof of this theorem will occupy the rest of this section.

Let us first prove part 1:

Lemma 1 Any dynamical r -matrix such that $r(\lambda) + r^{21}(\lambda) = \Omega$ is gauge-equivalent to a dynamical r -matrix $\tilde{r}(\lambda)$ such that $\tilde{r}(0) \in \frac{\Omega}{2} + (\Lambda^2(\mathfrak{l}'))^{\mathfrak{l}}$.

Proof: Let $\bar{\eta} \in \mathfrak{l} \otimes \mathfrak{g}^{\mathfrak{l}}$ be such that $r(0) - \bar{\eta} + \bar{\eta}^{21} \in \frac{\Omega}{2} + \Lambda^2(\mathfrak{l}')$. There exists a function $g : D \rightarrow G^L$ such that $g^{-1}dg(0) = \eta$ (see [EV], Lemma 1.3). It is easy to see that $\tilde{r} = r^g$ satisfies the desired conditions.

□

By Theorem 2, part 1. is proved. Let us now prove part 2. We will interpret the CDYBE (10) as a consistent system of differential equations defined on M_{Ω} .

For $s \in M_{\Omega}$, (10) is equivalent to

$$(\pi \otimes 1 \otimes 1) \text{Alt}(ds) = -(\pi \otimes 1 \otimes 1) (CYB(s) + \frac{1}{4} CYB(\Omega)).$$

This reduces to

$$ds = -(\pi \otimes 1 \otimes 1) ([s^{12}, s^{13}] + \frac{1}{4} CYB(\Omega)), \quad (12)$$

or, in coordinates (x_i) , where (x_i) is a basis of \mathfrak{l} ,

$$\frac{\partial s}{\partial x_i} = -(x_i \otimes 1 \otimes 1) ([s^{12}, s^{13}] + \frac{1}{4} CYB(\Omega)).$$

Lemma 2 The system (12) is consistent.

Proof: Set $X : M_\Omega \rightarrow \mathfrak{l} \otimes \mathfrak{g} \otimes \mathfrak{g}$, $s \mapsto (\pi \otimes 1 \otimes 1)([s^{12}, s^{13}] + \frac{1}{4}CYB(\Omega))$. By definition, the curvature of (12) is given by

$$\begin{aligned}
& \sum_{i,j} x_i \otimes x_j \otimes \left(\frac{\partial^2 s}{\partial x_i \partial x_j} - \frac{\partial^2 s}{\partial x_j \partial x_i} \right) \\
&= (\pi \otimes \pi \otimes 1 \otimes 1) \left(\{[s^{23}, [s^{12}, s^{14}]] + [s^{23}, \frac{1}{4}CYB(\Omega)^{124}] \right. \\
&\quad + [[s^{12}, s^{13}], s^{24}] + [\frac{1}{4}CYB(\Omega)^{123}, s^{24}] \} \\
&\quad - \{[s^{13}, [s^{21}, s^{24}]] + [s^{13}, \frac{1}{4}CYB(\Omega)^{214}] \\
&\quad + [[s^{21}, s^{23}], s^{14}] + [\frac{1}{4}CYB(\Omega)^{213}, s^{14}] \} \Big) \\
&= (\pi \otimes \pi \otimes 1 \otimes 1) \left(\{[s^{23}, [s^{12}, s^{14}]] + [[s^{12}, s^{13}], s^{24}] - [s^{13}, [s^{21}, s^{24}]] - [[s^{21}, s^{23}], s^{14}]] \} \right. \\
&\quad \left. + \frac{1}{4} \{[s^{13} + s^{23}, CYB(\Omega)^{124}] - [s^{14} + s^{24}, CYB(\Omega)^{123}]\} \right).
\end{aligned}$$

By the Jacobi identity,

$$[s^{23}, [s^{12}, s^{14}]] = [[s^{21}, s^{23}], s^{14}], \quad [[s^{12}, s^{13}], s^{24}] = [s^{13}, [s^{21}, s^{24}]].$$

By \mathfrak{g} -invariance of $CYB(\Omega)$, we have

$$\begin{aligned}
[s^{13} + s^{23}, CYB(\Omega)^{124}] &= [s^{34}, CYB(\Omega)^{124}], \\
[s^{14} + s^{24}, CYB(\Omega)^{123}] &= -[s^{34}, CYB(\Omega)^{123}].
\end{aligned}$$

Overall, we have the following expression for the curvature of (12):

$$\frac{1}{4}(\pi \otimes \pi \otimes 1 \otimes 1)([CYB(\Omega)^{123} + CYB(\Omega)^{124}, s^{34}] = \frac{1}{4}[(\pi \otimes \pi \otimes 1)CYB(\Omega), s]$$

But (9) and the fact that \mathfrak{l}' is admissible imply that $(\pi \otimes \pi \otimes 1)CYB(\Omega) = 0$. Thus, (12) is consistent.

□

Lemma 3 *The system (12) is defined on M_Ω , i.e the vector fields defined by (12) are tangent to M_Ω .*

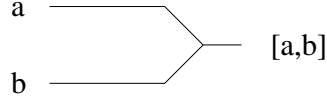
Proof: Let $x^* \in \mathfrak{l}^* \xrightarrow{\pi^*} \mathfrak{g}^*$, and set $h = (x^* \otimes 1 \otimes 1)([s^{12}, s^{13}] + \frac{1}{4}CYB(\Omega))$. Since $s \in \Lambda^2(\mathfrak{l}')$ we have $(x^* \otimes 1 \otimes 1)[s^{12}, s^{13}] \in \Lambda^2(\mathfrak{l}')$. Moreover, the admissibility of \mathfrak{l}' and (9) together imply that $(x^* \otimes 1 \otimes 1)(CYB(\Omega)) \in (\Lambda^2 \mathfrak{l}')^\mathfrak{l}$ since $[\mathfrak{l} \otimes 1, S^2 z_0(\mathfrak{g}^\mathfrak{l})] = 0$. Thus $h \in \Lambda^2 \mathfrak{l}'$.

To conclude the proof of Lemma 3 and Theorem 3, we now show that

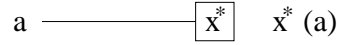
$$\begin{aligned}
& [s^{12}, h^{13}] + [s^{12}, h^{23}] + [s^{13}, h^{23}] \\
& + [h^{12}, s^{13}] + [h^{12}, s^{23}] + [h^{13}, s^{23}] \in \text{Alt}(\mathfrak{l} \otimes \mathfrak{g} \otimes \mathfrak{g}).
\end{aligned} \tag{13}$$

To make the presentation more clear, we will use the pictorial technique to represent expressions and make computations: we associate to each morphism

from a n -tensor to a m -tensor a diagram in the following way: the operation of taking the commutator is represented by



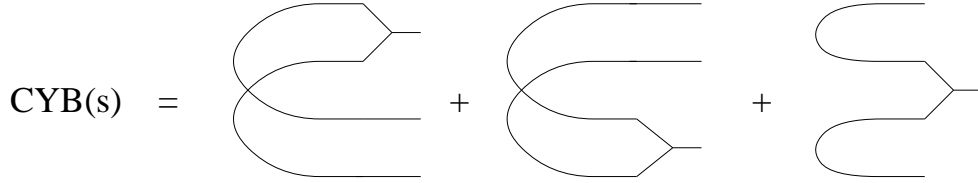
Applying a linear form x^* will be denoted by



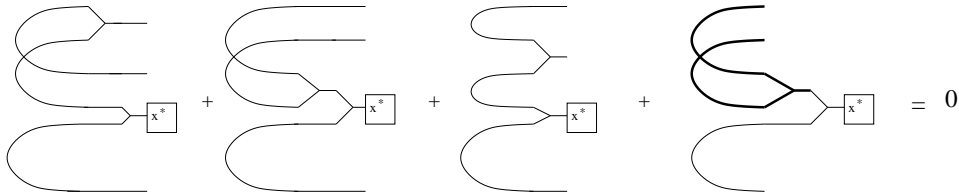
Finally, we will represent s and $\frac{\Omega}{2}$, which can be thought of as maps from a 0-tensor to a 2-tensor, by



For instance,



Lemma 4 *We have $x^{*(3)}[CYB(s + \frac{\Omega}{2})^{123}, s^{34}] \in \text{Alt}(\mathfrak{l} \otimes \mathfrak{g} \otimes \mathfrak{g})$ or, in pictures (modulo $\text{Alt}(\mathfrak{l} \otimes \mathfrak{g} \otimes \mathfrak{g})$)*



Proof: Recall that $CYB(s + \frac{\Omega}{2}) \in \text{Alt}(\mathfrak{l} \otimes \mathfrak{g} \otimes \mathfrak{g})$. Thus the only part of the above expression which can lie outside of $\text{Alt}(\mathfrak{l} \otimes \mathfrak{g} \otimes \mathfrak{g})$ is obtained from the $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{l}$ -part of $CYB(s)$. But if $y \in \mathfrak{l}$,

$$(x^* \otimes 1)[y \otimes 1, s] = -(x^* \otimes 1)[1 \otimes y, s]$$

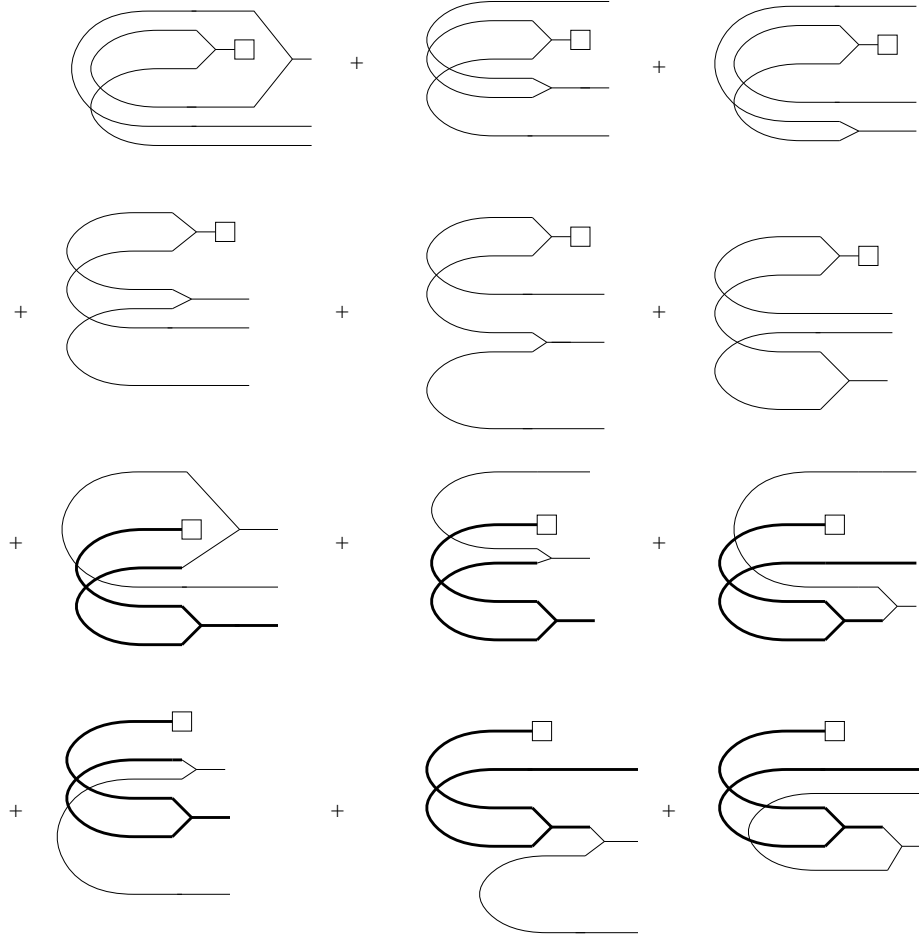
by \mathfrak{l} -invariance of s . This last expression is zero since $s \in (\Lambda^2(\mathfrak{l}'))^{\mathfrak{l}}$. Lemma 4 is proved.

□

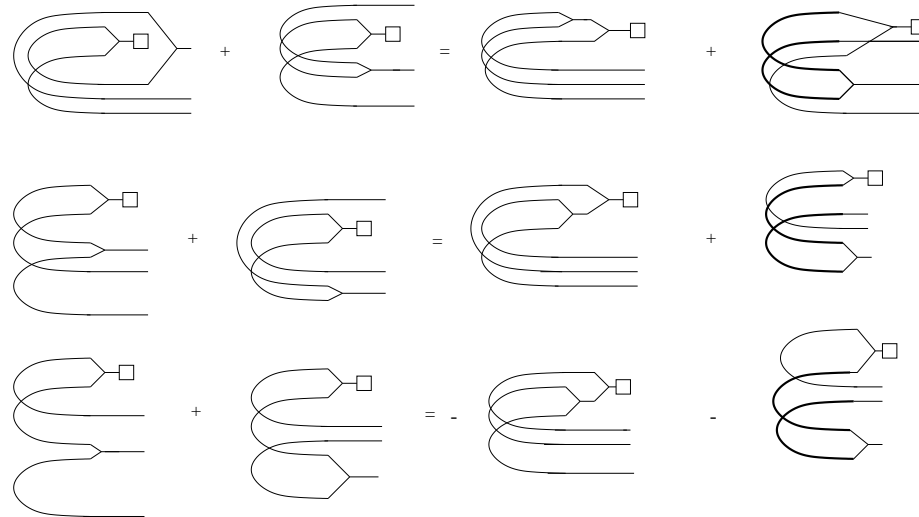
It is clear how to generalize Lemma 4 to other expressions of the form

$$x^{*(k)}[CYB(s + \frac{\Omega}{2})^{123}, s^{k4}].$$

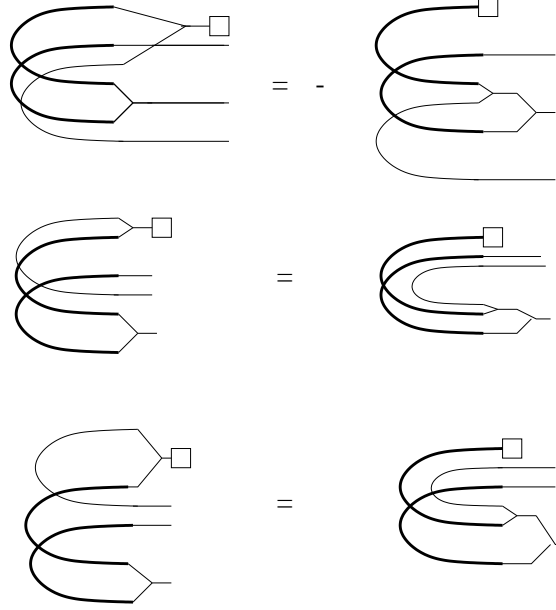
Now, (13) can be drawn as



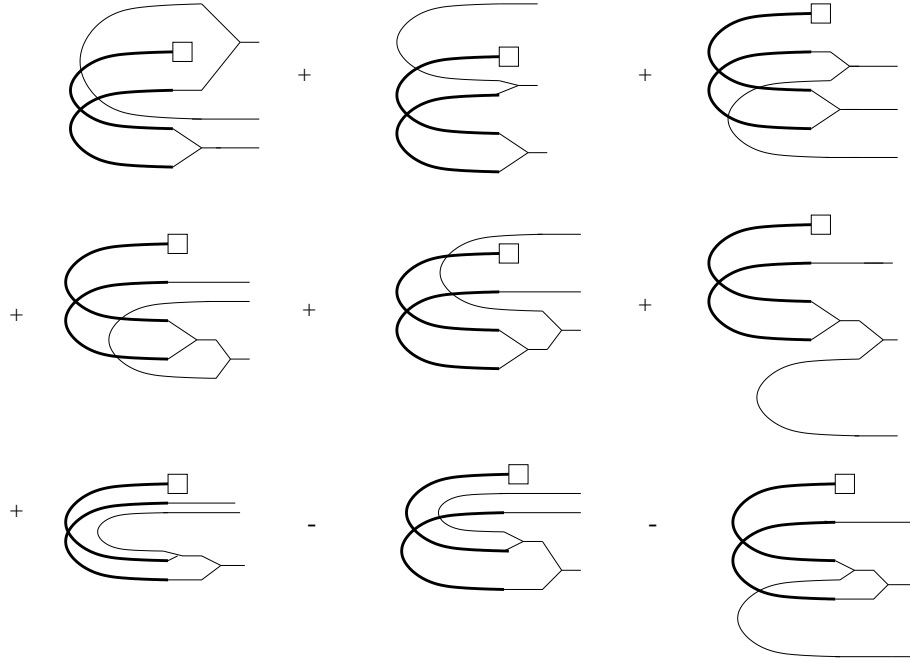
but by Lemma (4) we have, modulo $\text{Alt}(\mathfrak{l} \otimes \mathfrak{g} \otimes \mathfrak{g})$,



It is easy to check that the sum of the terms of type $[CYB(s), s]$ in this last expression is zero by the Jacobi identity. Moreover, by \mathfrak{g} -invariance of Ω , we have



Thus, modulo $\text{Alt}(\mathfrak{l} \otimes \mathfrak{g} \otimes \mathfrak{g})$, (13) reduces to



The sums of terms in each column is zero by Jacobi Identity. This concludes the proof of Theorem 3.

□

5 Classification of dynamical r-matrices

Let \mathfrak{g} be a simple algebra. In that case, (8) becomes

$$r(\lambda) + r^{21}(\lambda) = \epsilon \Omega. \quad (14)$$

We will classify all solutions of equations (6,7,14) when $\epsilon \neq 0$ and when \mathfrak{l} contains a semisimple regular element. In particular, in this case, the centralizer \mathfrak{h} of \mathfrak{l} is the unique Cartan subalgebra containing \mathfrak{l} . Notice that we can assume that $\epsilon = 1$ (since the assignment $r(\lambda) \rightarrow \epsilon r(\epsilon\lambda)$ is a gauge transformation of (7)). We can also assume that the restriction of (\cdot, \cdot) to \mathfrak{l} is nondegenerate. Indeed, for any dynamical r-matrix, we can replace \mathfrak{l} by the largest subspace of \mathfrak{h} for which r is invariant, and such a subspace is real. Let \mathfrak{h}_0 be the orthogonal complement of \mathfrak{l} in \mathfrak{h} and let $i : \mathfrak{l} \hookrightarrow \mathfrak{h}$ be the inclusion map. We will also write (\cdot, \cdot) for the induced bracket on \mathfrak{l}^* . Let $\Omega_{\mathfrak{h}_0}$ denote the Casimir element of the restriction of (\cdot, \cdot) to \mathfrak{h}_0 .

5.1 Statement of the theorem

Let $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ be a polarization of \mathfrak{g} .

Definition: A generalized Belavin-Drinfeld triple is a triple $(\Gamma_1, \Gamma_2, \tau)$ where $\Gamma_1, \Gamma_2 \subset \Pi$, and $\tau : \Gamma_1 \xrightarrow{\sim} \Gamma_2$ is a norm-preserving bijection.

In other terms, in a generalized Belavin-Drinfeld triple, we drop the nilpotency condition. We will say that a generalized Belavin-Drinfeld triple is \mathfrak{l} -graded if τ preserves the decomposition of \mathfrak{g} in \mathfrak{l} -weight spaces. If $(\Gamma_1, \Gamma_2, \tau)$ is a generalized Belavin-Drinfeld triple, we will denote by Γ_3 the largest subset of $\Gamma_1 \cap \Gamma_2$ which is stable under τ , and $\tilde{\Gamma}_1 = \Gamma_1 \setminus \Gamma_3$, $\tilde{\Gamma}_2 = \Gamma_2 \setminus \Gamma_3$. It is clear that $(\tilde{\Gamma}_1, \tilde{\Gamma}_2, \tau)$ is a Belavin-Drinfeld triple. As before, for each choice of Chevalley generators $(e_\alpha, f_\alpha, h_\alpha)_{\alpha \in \Gamma_i}$, the map τ induces isomorphisms $\mathfrak{g}_{\tilde{\Gamma}_1} \rightarrow \mathfrak{g}_{\tilde{\Gamma}_2}$ and $\tau : \mathfrak{g}_{\Gamma_3} \rightarrow \mathfrak{g}_{\Gamma_3}$.

For $\lambda \in \mathfrak{l}^*$, consider the map:

$$K(\lambda) : \mathfrak{n}_+(\Gamma_1) \rightarrow \mathfrak{n}_+(\Gamma_2)$$

$$e_\alpha \mapsto \frac{1}{2}e_\alpha + e^{-(\alpha, \lambda)} \frac{\tau}{1 - e^{-(\alpha, \lambda)\tau}}(e_\alpha).$$

Notice that we have

$$K(\lambda)(e_\alpha) = \frac{1}{2}e_\alpha + \sum_{n>0} e^{-n(\alpha, \lambda)} \tau^n(e_\alpha).$$

This sum is finite for $\alpha \notin \langle \Gamma_3 \rangle$.

Theorem 4 *Let \mathfrak{g} be a simple Lie algebra with nondegenerate invariant bilinear form (\cdot, \cdot) , $\mathfrak{l} \subset \mathfrak{g}$ a commutative subalgebra containing a regular semisimple element on which (\cdot, \cdot) is nondegenerate, \mathfrak{h} the Cartan subalgebra containing \mathfrak{l} and \mathfrak{h}_0 the orthogonal complement of \mathfrak{l} in \mathfrak{h} . Then*

1. *Any dynamical r-matrix is gauge-equivalent to a dynamical r-matrix \tilde{r} such that*

$$\tilde{r}(\lambda) - \tilde{r}(\lambda)^{21} \in (\mathfrak{l}^\perp)^{\otimes 2} = \left(\bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha \oplus \mathfrak{h}_0 \right)^{\otimes 2}. \quad (15)$$

2. Let $(\Gamma_1, \Gamma_2, \tau)$ be an \mathfrak{l} -graded generalized Belavin-Drinfeld triple and let $(e_\alpha, f_\alpha, h_\alpha)_{\Gamma_i}$ be a choice of Chevalley generators. Let $r_{\mathfrak{h}_0, \mathfrak{h}_0} \in \mathfrak{h}_0 \otimes \mathfrak{h}_0$ satisfy the equation

$$(\tau(\alpha) \otimes 1)r_{\mathfrak{h}_0, \mathfrak{h}_0} + (1 \otimes \alpha)r_{\mathfrak{h}_0, \mathfrak{h}_0} = \frac{1}{2}((\alpha + \tau(\alpha)) \otimes 1)\Omega_{\mathfrak{h}_0}. \quad (16)$$

Then

$$r(\lambda) = \frac{1}{2}\Omega + r_{\mathfrak{h}_0, \mathfrak{h}_0} + \sum_{\alpha \in \langle \Gamma_1 \rangle \cap \Delta_+} K(\lambda)(e_\alpha) \wedge e_{-\alpha} + \sum_{\alpha \in \Delta_+, \alpha \notin \langle \Gamma_1 \rangle} \frac{1}{2}e_\alpha \wedge e_{-\alpha}$$

is a solution the CDYBE satisfying (15).

3. Any solution of the CDYBE satisfying (15) is of the above type for a suitable polarization of \mathfrak{g} .

The proof of this theorem will occupy the rest of this section. Our methods are greatly inspired by the paper [BD]. Notice that 1. follows from Theorem 3, but we will describe the gauge transformations explicitly in this case.

Notations: Let $\Delta \subset \mathfrak{h}^*$ be the root system of \mathfrak{g} with respect to \mathfrak{h} and set $\Delta_{\mathfrak{l}} = i^*(\Delta) \subset \mathfrak{l}^*$. We will denote by $\mathfrak{g}_{\overline{\alpha}}$ the weight subspace associated to $\overline{\alpha} = i^*(\alpha) \in \Delta_{\mathfrak{l}}$, and we set $\mathfrak{g}_{\overline{0}} = \mathfrak{h}_0$. It is clear that

$$\mathfrak{g}_{\overline{\alpha}} = \bigoplus_{\beta \in \Delta, i^*(\beta) = \overline{\alpha}} \mathfrak{g}_{\beta}$$

In particular, $(,)$ is a pairing $\mathfrak{g}_{\overline{\alpha}} \times \mathfrak{g}_{-\overline{\alpha}} \rightarrow \mathbb{C}$.

A vector space $V \subset \mathfrak{g}$ will be called \mathfrak{h} -graded (resp. \mathfrak{l} -graded) if it is an \mathfrak{h} -submodule (resp. \mathfrak{l} -submodule) of \mathfrak{g} . Finally, let $\Omega' \in (\mathfrak{l}^\perp)^{\otimes 2}$ denote the Casimir (inverse element) of the restriction of $(,)$ to $\mathfrak{l}^\perp = \mathfrak{h}_0 \oplus \mathfrak{g}_{\overline{\alpha}}$.

Now let $r : \mathfrak{l}^* \supset D \rightarrow (\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{l}}$ be a formal power series satisfying (14) (with $\epsilon = 1$). By (6), we can write

$$r(\lambda) = \frac{1}{2}\Omega + r_{\mathfrak{l}, \mathfrak{l}}(\lambda) + r_{\mathfrak{l}, \mathfrak{h}_0}(\lambda) + r_{\mathfrak{h}_0, \mathfrak{l}}(\lambda) + (\varphi(\lambda) \otimes 1)\Omega', \quad (17)$$

where $r_{\mathfrak{l}, \mathfrak{l}}(\lambda) \in \mathfrak{l} \otimes \mathfrak{l}$, $r_{\mathfrak{l}, \mathfrak{h}_0}(\lambda) \in \mathfrak{l} \otimes \mathfrak{h}_0$, $r_{\mathfrak{h}_0, \mathfrak{l}}(\lambda) \in \mathfrak{h}_0 \otimes \mathfrak{l}$ and where $\varphi(\lambda) \in \text{End}(\mathfrak{h}_0 \oplus \mathfrak{g}_{\overline{\alpha}})$ is a sum of maps $\varphi_{\overline{\alpha}}(\lambda) \in \text{End}(\mathfrak{g}_{\overline{\alpha}})$. By the unitarity condition, $r_{\mathfrak{l}, \mathfrak{l}}(\lambda) \in \Lambda^2 \mathfrak{l}$, $r_{\mathfrak{l}, \mathfrak{h}_0}(\lambda) = -r_{\mathfrak{h}_0, \mathfrak{l}}^{21}(\lambda)$ and $\varphi_{-\overline{\alpha}}(\lambda) = -\varphi_{\overline{\alpha}}^*(\lambda)$.

With these notations, the CDYBE splits into 4 components: the $\mathfrak{l} \otimes \mathfrak{l} \otimes \mathfrak{l}$ -part, the $\mathfrak{l} \otimes \mathfrak{l} \otimes \mathfrak{h}_0$ -part, the $\mathfrak{l} \otimes \mathfrak{g}_{\overline{\alpha}} \otimes \mathfrak{g}_{-\overline{\alpha}}$ -part and the $\mathfrak{g}_{\overline{\alpha}} \otimes \mathfrak{g}_{\overline{\beta}} \otimes \mathfrak{g}_{\overline{\gamma}}$ -part where $\overline{\alpha} + \overline{\beta} + \overline{\gamma} = 0$.

- The $\mathfrak{l} \otimes \mathfrak{l} \otimes \mathfrak{l}$ -part: let us set $r_{\mathfrak{l}, \mathfrak{l}} = \sum_{i,j} C_{i,j}(\lambda) x_i \otimes x_j$. This part of the CDYBE can then be written:

$$\frac{\partial C_{j,k}}{\partial x_i} + \frac{\partial C_{k,i}}{\partial x_j} + \frac{\partial C_{i,j}}{\partial x_k} = 0 \quad \forall i, j, k \quad (18)$$

and says that $\sum_{i,j} C_{i,j} dx_i \wedge dx_j$ is a closed 2-form.

- The $\mathfrak{l} \otimes \mathfrak{l} \otimes \mathfrak{h}_0$ -part: let us set $r_{\mathfrak{l}, \mathfrak{h}_0} = \sum_{i,j} D_{i,j}(\lambda) x_i \otimes y_j$ for some basis (y_j) of \mathfrak{h}_0 . This part of the CDYBE is

$$\frac{\partial D_{i,j}}{\partial x_k} = \frac{\partial D_{k,j}}{\partial x_i} \quad \forall i, k, j \quad (19)$$

and says that for any j , $\sum_i D_{i,j}(\lambda) dx_i$ is a closed 1-form.

Since r is defined on a polydisc, the above forms are exact. Let $f : D \rightarrow \mathfrak{h}_0$ be such that $df(\lambda) = \sum_i D_{i,j}(\lambda) dx_i \otimes y_j$ and let ξ be a 1-form on D such that $d\xi = \sum_{i,j} C_{i,j} dx_i \wedge dx_j$. Then ξ defines a function $\bar{\xi} : D \rightarrow \mathfrak{l}$. The gauge transformation which should be applied to r to make it satisfy (15) is easily seen to be the following:

$$r(\lambda) \mapsto r(\lambda)^g = \frac{1}{2}\Omega + (e^{-ad f(\lambda)} \varphi(\lambda) e^{ad f(\lambda)} \otimes 1) \Omega'$$

where $g(\lambda) = e^{f(\lambda)} e^{-\bar{\xi}(\lambda)}$.

Thus, we can assume that $r_{\mathfrak{l}, \mathfrak{l}} = r_{\mathfrak{l}, \mathfrak{h}_0} = 0$, in which case the remaining components of the CDYBE can be written in the following way:

- The $\mathfrak{l} \otimes \mathfrak{g}_{\bar{\alpha}} \otimes \mathfrak{g}_{-\bar{\alpha}}$ -part:

$$d\varphi_{\bar{\alpha}} + (\varphi_{\bar{\alpha}}^2 - \frac{1}{4}) dh_{\bar{\alpha}} = 0. \quad (20)$$

In particular, $r_{\mathfrak{h}_0, \mathfrak{h}_0} \in \Lambda^2 \mathfrak{h}_0$ is constant.

- The $\mathfrak{g}_{\bar{\alpha}} \otimes \mathfrak{g}_{\bar{\beta}} \otimes \mathfrak{g}_{\bar{\gamma}}$ -part where $\bar{\alpha} + \bar{\beta} + \bar{\gamma} = 0$:

$$\Lambda(\varphi_{\bar{\alpha}} \otimes \varphi_{\bar{\beta}} \otimes 1 + \varphi_{\bar{\alpha}} \otimes 1 \otimes \varphi_{\bar{\gamma}} + 1 \otimes \varphi_{\bar{\beta}} \otimes \varphi_{\bar{\gamma}} + \frac{1}{4} Id) = 0 \quad (21)$$

where $\Lambda : \mathfrak{g}_{\bar{\alpha}} \otimes \mathfrak{g}_{\bar{\beta}} \otimes \mathfrak{g}_{\bar{\gamma}} \rightarrow \mathbb{C}$, $x \otimes y \otimes z \mapsto ([x, y], z)$.

This set of equations is sufficient by skew-symmetry of the CDYBE.

5.2 The Cayley transform

Let us set $A_{\pm} = \text{Im}(\varphi(\lambda) \pm \frac{1}{2})$, $I_{\pm} = \text{Ker}(\varphi(\lambda) \mp \frac{1}{2})$. Notice that, by (20), A_{\pm} and I_{\pm} are indeed independent of λ . Furthermore, A_{\pm} , I_{\pm} are \mathfrak{l} -graded by the weight-zero condition, $I_{\pm} \subset A_{\pm}$ and $A_{\pm} = I_{\pm}^{\perp}$ by the unitarity condition. Notice also that $A_+ + A_- \oplus \mathfrak{l} = \mathfrak{g}$. Now consider

$$\psi(\lambda) = \frac{\varphi - \frac{1}{2}}{\varphi + \frac{1}{2}} : A_+/I_+ \rightarrow A_-/I_-.$$

Extend $\psi(\lambda)$ to $\psi(\lambda) : \mathfrak{l} \oplus A_+/I_+ \rightarrow \mathfrak{l} \oplus A_-/I_-$ by setting $\psi|_{\mathfrak{l}} = Id$. It is clear that ψ is a well-defined linear isomorphism. The following proposition is crucial:

Proposition 2 *The maps $\varphi_{\bar{\alpha}}$ satisfy (20,21) if and only if the following hold:*

(i) $A_{\pm} \oplus \mathfrak{l}$ is a subalgebra of \mathfrak{g} and $I_{\pm} \oplus \mathfrak{l}$ is an ideal of $A_{\pm} \oplus \mathfrak{l}$.

(ii) there exists a (constant) map $\psi_0 : \mathfrak{l} \oplus A_+/I_+ \rightarrow \mathfrak{l} \oplus A_-/I_-$ such that

$$\psi(\lambda)|_{\mathfrak{g}_{\bar{\alpha}}} = e^{-(\bar{\alpha}, \lambda)} \psi_0|_{\mathfrak{g}_{\bar{\alpha}}}.$$

(iii) The map ψ_0 is a Lie algebra map:

$$[\psi_0(x), \psi_0(y)] = \psi_0[x, y]. \quad (22)$$

Proof: Assume that φ satisfies (20,21) and let $a \in \mathfrak{g}_{\bar{\alpha}}$, $b \in \mathfrak{g}_{\bar{\beta}}$, $c \in \mathfrak{g}_{\bar{\gamma}}$ with $\bar{\alpha} + \bar{\beta} + \bar{\gamma} = 0$. From (21), we have

$$\begin{aligned} & ([(\varphi_{\bar{\alpha}} + \frac{1}{2})a, (\varphi_{\bar{\beta}} + \frac{1}{2})b], c) + ([a, (\varphi_{\bar{\beta}} + \frac{1}{2})b], (\varphi_{\bar{\gamma}} - \frac{1}{2})c) \\ & + ([(\varphi_{\bar{\alpha}} - \frac{1}{2})a, b], (\varphi_{\bar{\gamma}} - \frac{1}{2})c) = 0. \end{aligned}$$

Since $\varphi_{\bar{\gamma}} = -\varphi_{-\bar{\gamma}}^*$, and $(,)$ is a nondegenerate pairing $\mathfrak{g}_{\bar{\gamma}} \otimes \mathfrak{g}_{-\bar{\gamma}} \rightarrow \mathbb{C}$, this implies that $A_+ \oplus \mathfrak{l}$ is a Lie subalgebra of \mathfrak{g} . Note that the term in \mathfrak{l} is necessary here since $[\mathfrak{g}_{\bar{\alpha}}, \mathfrak{g}_{-\bar{\alpha}}] \not\subset \mathfrak{g}_{\bar{0}} = \mathfrak{h}_0$, but is not consequential as A_+ is \mathfrak{l} -graded. The second claim of (i) follows from the relation

$$\begin{aligned} & ([(\varphi_{\bar{\alpha}} - \frac{1}{2})a, (\varphi_{\bar{\beta}} - \frac{1}{2})b], c) + ([a, (\varphi_{\bar{\beta}} + \frac{1}{2})b], (\varphi_{\bar{\gamma}} + \frac{1}{2})c) \\ & + ([(\varphi_{\bar{\alpha}} - \frac{1}{2})a, b], (\varphi_{\bar{\gamma}} + \frac{1}{2})c) = 0. \end{aligned}$$

The proof is the same for A_- and I_- . The equivalence of (ii) and (20) follows from the equality

$$\begin{aligned} d\psi|_{\mathfrak{g}_{\bar{\alpha}}} &= \frac{d\varphi_{\alpha}(\varphi_{\alpha} + \frac{1}{2}) - (\varphi_{\alpha} - \frac{1}{2})d\varphi_{\alpha}}{(\varphi_{\alpha} + \frac{1}{2})^2} \\ &= -\frac{(\varphi_{\alpha}^2 - \frac{1}{4})}{(\varphi_{\alpha} + \frac{1}{2})^2} dh_{\bar{\alpha}} \\ &= -(\bar{\alpha}, \lambda)\psi|_{\mathfrak{g}_{\bar{\alpha}}}. \end{aligned}$$

where we used (20). Finally it follows from (21) that

$$(\varphi_{\bar{\alpha}+\bar{\beta}} - \frac{1}{2})[(\varphi_{\bar{\alpha}} + \frac{1}{2})a, (\varphi_{\bar{\beta}} + \frac{1}{2})b] = (\varphi_{\bar{\alpha}+\bar{\beta}} + \frac{1}{2})[(\varphi_{\bar{\alpha}} - \frac{1}{2})a, (\varphi_{\bar{\beta}} - \frac{1}{2})b].$$

This implies (iii).

Conversely, if (i-iii) are satisfied then for any $x \in \mathfrak{g}_{\bar{\alpha}}$, $y \in \mathfrak{g}_{\bar{\beta}}$ ($\bar{\alpha} + \bar{\beta} \neq 0$) there exist $z \in \mathfrak{g}_{\bar{\alpha}+\bar{\beta}}$ such that

$$[(\varphi_{\bar{\alpha}} - \frac{1}{2})x, (\varphi_{\bar{\beta}} - \frac{1}{2})y] = (\varphi_{\bar{\alpha}+\bar{\beta}} - \frac{1}{2})z.$$

Since ψ is a Lie algebra map, $[(\varphi_{\bar{\alpha}} + \frac{1}{2})x, (\varphi_{\bar{\beta}} + \frac{1}{2})y] - (\varphi_{\bar{\alpha}+\bar{\beta}} + \frac{1}{2})z \in \text{Ker}(\varphi_{\bar{\alpha}+\bar{\beta}} - \frac{1}{2})$. Subtracting, we obtain

$$[(\varphi_{\bar{\alpha}} + \frac{1}{2})x, y] + [x, (\varphi_{\bar{\beta}} + \frac{1}{2})y] - [x, y] - z \in \text{Ker}(\varphi_{\bar{\alpha}+\bar{\beta}} - \frac{1}{2}).$$

Applying $(\varphi - \frac{1}{2})$ and dropping the indices, we have

$$(\varphi - \frac{1}{2})\left([\left(\varphi + \frac{1}{2}\right)x, y] + [x, \left(\varphi + \frac{1}{2}\right)y] - [x, y]\right) = \left[\left(\varphi - \frac{1}{2}\right)x, \left(\varphi - \frac{1}{2}\right)y\right].$$

Thus,

$$\left[\left(\varphi + \frac{1}{2}\right)x, \left(\varphi + \frac{1}{2}\right)y\right] - \left(\varphi + \frac{1}{2}\right)\left([\left(\varphi - \frac{1}{2}\right)x, y] + [x, \left(\varphi + \frac{1}{2}\right)y]\right) = 0.$$

which is equivalent to (21). □

We will call the triple (A_+, A_-, ψ_0) the Cayley transform of φ . We are now reduced to the classification of all triples satisfying (i-iii) and which arise as a Cayley transform (Cayley triples).

5.3 Classification of Cayley triples

Let (A_+, A_-, ψ_0) be a Cayley triple. If $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ is a polarization of \mathfrak{g} and $\Gamma \subset \Pi(\mathfrak{n}_+)$ we will denote by \mathfrak{q}_Γ^+ (resp. \mathfrak{q}_Γ^-) the subalgebra generated by \mathfrak{n}_+ and $\mathfrak{g}_{-\alpha}$, $\alpha \in \Gamma$ (resp. generated by \mathfrak{n}_- and \mathfrak{g}_α , $\alpha \in \Gamma$). We denote by $\mathfrak{p}_\Gamma^\pm = \mathfrak{h} + \mathfrak{q}_\Gamma^\pm$ the parabolic subalgebras associated to Γ .

Proposition 3 *There exists a polarization $\mathfrak{g} = \mathfrak{n}_+^1 \oplus \mathfrak{h} \oplus \mathfrak{n}_-^1$, two subsets $\Gamma_+, \Gamma_- \subset \Pi(\mathfrak{n}_+^1)$ and two vector spaces $V_+, V_- \subset \mathfrak{h}$ with $V_\pm^\perp \subset V_\pm$ such that*

$$\mathfrak{l} \oplus A_+ = \mathfrak{q}_{\Gamma_+}^+ \oplus V_+, \quad \mathfrak{l} \oplus A_- = \mathfrak{q}_{\Gamma_-}^- \oplus V_-$$

Proof: Notice that $(\mathfrak{l} \oplus A_+)^{\perp} = I_+ \subset \mathfrak{l} \oplus A_+$. It is known, (c.f [Bou, chap.VIII, §10, Thm. 1] or [BD]), that this implies that $\mathfrak{l} \oplus A_+ = \tilde{\mathfrak{q}}_\Gamma^+ \oplus \tilde{V}_+$ for some polarization $\mathfrak{g} = \mathfrak{n}'_+ \oplus \mathfrak{h}' \oplus \mathfrak{n}'_-$. Similarly, $\mathfrak{l} \oplus A_- = \tilde{\mathfrak{q}}_{\Gamma'}^- \oplus \tilde{V}_-$ for some polarization $\mathfrak{g} = \mathfrak{n}''_+ \oplus \mathfrak{h}'' \oplus \mathfrak{n}''_-$. Moreover, \mathfrak{l} acts semisimply on A_\pm so $\mathfrak{l} \subset \mathfrak{h}'$, $\mathfrak{l} \subset \mathfrak{h}''$. But \mathfrak{l} contains a regular element, thus $\mathfrak{l} = \mathfrak{h}' = \mathfrak{h}''$. Proposition 3 is now an easy consequence of the following lemma:

Lemma 5 *Let \mathfrak{g} be a simple Lie algebra and \mathfrak{h} a Cartan subalgebra. Let \mathfrak{a}_1 and \mathfrak{a}_2 be two parabolic subalgebras containing \mathfrak{h} such that $\mathfrak{a}_1 + \mathfrak{a}_2 = \mathfrak{g}$. Then there exists a polarization $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ and $\Gamma_+, \Gamma_- \subset \Pi$ such that $\mathfrak{a}_1 = \mathfrak{p}_{\Gamma_+}^+$ and $\mathfrak{a}_2 = \mathfrak{p}_{\Gamma_-}^-$.*

Proof: Let $\mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ be a polarization of \mathfrak{g} such that $\mathfrak{b}_+ \subset \mathfrak{a}_1$ and for which $\dim(\mathfrak{n}_+ \cap \mathfrak{a}_2)$ is minimal. We claim that $\mathfrak{b}_- \subset \mathfrak{a}_2$. Suppose on the contrary that there exists a simple root $\alpha \in \Pi$ such that $\mathfrak{g}_{-\alpha} \not\subset \mathfrak{a}_2$. Then $\mathfrak{g}_{-\alpha} \subset \mathfrak{a}_1$ since $\mathfrak{a}_1 + \mathfrak{a}_2 = \mathfrak{g}$ and $\mathfrak{g}_\alpha \subset \mathfrak{a}_2$ since \mathfrak{a}_2 is parabolic. But then $s_\alpha \mathfrak{n}_+ \oplus \mathfrak{h} \oplus s_\alpha \mathfrak{n}_-$ is a polarization of \mathfrak{g} for which $s_\alpha \mathfrak{b}_+ \subset \mathfrak{a}_1$ and $\dim(s_\alpha \mathfrak{n}_+ \cap \mathfrak{a}_2) < \dim(\mathfrak{n}_+ \cap \mathfrak{a}_2)$. Contradiction. □

In particular, A_\pm, I_\pm are all \mathfrak{h} -graded and

$$\begin{aligned} I_+ &= (\mathfrak{q}_{\Gamma_+}^+ \oplus V_+)^{\perp} = \bigoplus_{\alpha \in \Delta_+ \setminus \langle \Gamma_+ \rangle} \mathfrak{g}_\alpha \oplus (V_+^{\perp} \cap \mathfrak{h}_0), \\ I_- &= (\mathfrak{q}_{\Gamma_-}^- \oplus V_-)^{\perp} = \bigoplus_{\alpha \in \Delta_- \setminus \langle \Gamma_- \rangle} \mathfrak{g}_\alpha \oplus (V_-^{\perp} \cap \mathfrak{h}_0). \end{aligned}$$

Thus $A_+/I_+ = \mathfrak{g}_{\Gamma_+} \oplus V_1$ and $A_-/I_- = \mathfrak{g}_{\Gamma_-} \oplus V_2$ for some suitable $V_1, V_2 \subset \mathfrak{h}_0$.

Let $L_{\pm\frac{1}{2}}(\lambda)$ be the generalized eigenspace of $\varphi(\lambda)$ associated to $\pm\frac{1}{2}$. Since φ is a solution of an ordinary differential equation with stationary points at $\frac{1}{2}, -\frac{1}{2}$, $L_{\pm\frac{1}{2}}(\lambda)$ is independent of λ and we will simply denote it by $L_{\pm\frac{1}{2}}$. Similarly, let L' be the sum of all other generalized eigenspaces so that $\mathfrak{g} = \mathfrak{l} \oplus L_{\frac{1}{2}} \oplus L' \oplus L_{-\frac{1}{2}}$.

Proposition 4 *There exists a polarization $\mathfrak{g} = \bar{\mathfrak{n}}_+ \oplus \mathfrak{h} \oplus \bar{\mathfrak{n}}_-$ and a subset $\Gamma_3 \subset \Pi(\bar{\mathfrak{n}}_+)$ such that $L_{\pm\frac{1}{2}} \subset \bar{\mathfrak{b}}_{\pm}$, $L' \subset \mathfrak{g}_{\Gamma_3} + \mathfrak{h}$ and $\varphi(\bar{\mathfrak{n}}_+) \subset \bar{\mathfrak{n}}_+$.*

Proof: We will construct a polarization satisfying the above conditions in several steps.

Lemma 6 *We have:*

- (i) $\mathfrak{l} \oplus L_{\pm\frac{1}{2}}$ is an \mathfrak{h} -graded solvable subalgebra,
- (ii) $\mathfrak{l} \oplus L'$ is an \mathfrak{h} -graded subalgebra,
- (iii) we have $[L_{\pm\frac{1}{2}}, L'] \subset \mathfrak{l} \oplus L_{\pm\frac{1}{2}}$.

Proof: this follows from the proofs of Lemma 12.3 and Theorem 12.6 in [BD].

Notice that $L_{\pm\frac{1}{2}} \not\subset \mathfrak{b}_{\pm}^1$ in general. We first construct a polarization $\mathfrak{g} = \mathfrak{n}_+^2 \oplus \mathfrak{h} \oplus \mathfrak{n}_-^2$ such that $L_{\pm\frac{1}{2}} \subset \mathfrak{b}_{\pm}^2$. We have $I_{\pm} \subset L_{\pm\frac{1}{2}}$. Notice that $L_{\frac{1}{2}} \cap \mathfrak{n}_-^1 \subset \mathfrak{g}_{\Gamma_+} \cap \mathfrak{g}_{\Gamma_-} = \mathfrak{g}_{\Gamma_+ \cap \Gamma_-}$ since $\mathfrak{n}_-^1 \subset (\mathfrak{g}_{\Gamma_-} \oplus I_-)$ and $L_{\frac{1}{2}}$ is solvable. Similarly, $L_{-\frac{1}{2}} \cap \mathfrak{n}_+^1 \subset \mathfrak{g}_{\Gamma_+ \cap \Gamma_-}$. Moreover, by Lemma 6, $\mathfrak{l} \oplus (L_{\frac{1}{2}} \cap \mathfrak{g}_{\Gamma_+ \cap \Gamma_-})$ and $\mathfrak{l} \oplus (L_{-\frac{1}{2}} \cap \mathfrak{g}_{\Gamma_+ \cap \Gamma_-})$ are disjoint, solvable, \mathfrak{h} -graded subalgebras. By lemma 5 it follows that there exists an element s of the group $W_{\Gamma_+ \cap \Gamma_-}$ such that

$$\mathfrak{l} \oplus (L_{\pm\frac{1}{2}} \cap \mathfrak{g}_{\Gamma_+ \cap \Gamma_-}) \subset s\mathfrak{b}_{\pm}^1.$$

Notice that s permutes elements of $\Delta^+ \setminus \langle \Gamma_+ \cap \Gamma_- \rangle$, leaving it globally unchanged. Thus, $\mathfrak{l} \oplus L_{\pm\frac{1}{2}} \subset s\mathfrak{b}_{\pm}^1$. Set $\mathfrak{n}_{\pm}^2 = s\mathfrak{n}_{\pm}^1$.

Now we construct a polarization of \mathfrak{g} satisfying the other conditions of proposition 4. Recall that $\mathfrak{l} \oplus L \subset \mathfrak{g}_{\Gamma_+ \cap \Gamma_-} + (V_1 \cap V_2)$. Thus

$$(L' \cap \mathfrak{n}_+^2) \oplus (L_{\frac{1}{2}} \cap \mathfrak{n}_+^2(\Gamma_+ \cap \Gamma_-)) = \mathfrak{n}_+^2(\Gamma_+ \cap \Gamma_-).$$

Since $[L', L_{\frac{1}{2}}] \subset \mathfrak{l} \oplus L_{\frac{1}{2}}$ by Lemma 6, (iii), $L_{\frac{1}{2}} \cap \mathfrak{n}_+^2(\Gamma_+ \cap \Gamma_-)$ is an ideal of $\mathfrak{n}_+^2(\Gamma_+ \cap \Gamma_-)$. But $L' \cap \mathfrak{n}_+^2$ is a subalgebra. It is easy to see that this implies that $L' \cap \mathfrak{n}_+^2$ is generated by a set of simple root subspaces of $\mathfrak{n}_+^2(\Gamma_+ \cap \Gamma_-)$, i.e

$L' \cap \mathfrak{n}_+^2 = \mathfrak{n}_+^2(\Gamma)$ for some $\Gamma \subset \Pi(\mathfrak{n}_+^2)$. Moreover, the restriction of (\cdot, \cdot) to L' is nondegenerate, hence $L' \cap \mathfrak{n}_-^2 = \mathfrak{n}_-^2(-\Gamma)$. Thus

$$\mathfrak{l} \oplus \mathfrak{g}_\Gamma \subset \mathfrak{l} \oplus L' \subset \mathfrak{l} \oplus \mathfrak{g}_\Gamma + (V_1 \cap V_2).$$

Since $\varphi(\lambda) + \frac{1}{2}$ is invertible in L' , $\psi(\lambda)$ is a well-defined operator $L' \rightarrow L'$, satisfying (22), and $\psi(\lambda)(\mathfrak{h}_0 \cap L') \subset \mathfrak{h}_0 \cap L'$. Now, \mathfrak{l} contains a regular element. Thus there exists a polarization of \mathfrak{g} compatible with the \mathfrak{l} -weight decomposition. This induces a polarization of \mathfrak{g}_Γ , compatible with the \mathfrak{l} -weight decomposition of \mathfrak{g}_Γ . Hence, there exists $s' \in W_\Gamma \subset W$ such that $\psi_0|_{\mathfrak{g}_\Gamma}$ is compatible with the polarization $s'\mathfrak{n}_+^2 \oplus \mathfrak{h} \oplus s'\mathfrak{n}_-^2$. Since s' leaves $\Delta_+ \setminus \langle \Gamma \rangle$ globally unchanged, the polarization $\mathfrak{g} = \bar{\mathfrak{n}}_+ \oplus \mathfrak{h} \oplus \bar{\mathfrak{n}}_-$ with $\bar{\mathfrak{n}}_\pm = s'\mathfrak{n}_\pm^2$ and $\Gamma_3 = s'\Gamma$ satisfies the requirements of proposition 4.

□

To sum up, we have shown that there exists a polarization $\mathfrak{g} = \bar{\mathfrak{n}}_+ \oplus \mathfrak{h} \oplus \bar{\mathfrak{n}}_-$, compatible with φ , subsets $\Gamma_1 = s'\Gamma_+$, $\Gamma_2 = s'\Gamma_-$ and $\Gamma_3 \subset \Pi(\bar{\mathfrak{n}}_+)$ such that $(A_+/I_+) \cap \mathfrak{n}_+ = \bar{\mathfrak{n}}_+(\Gamma_1)$, $A_- \cap \mathfrak{n}_+ = \bar{\mathfrak{n}}_+(\Gamma_2)$ and $L' \cap \mathfrak{n}_+ = \bar{\mathfrak{n}}_+(\Gamma_3)$.

The map ψ_0 now restricts to a Lie algebra isomorphism $\bar{\mathfrak{n}}_+(\Gamma_1) \rightarrow \bar{\mathfrak{n}}_+(\Gamma_2)$. This isomorphism maps weight spaces to weight spaces as ψ_0 preserves \mathfrak{h}_0 and φ is \mathfrak{l} -invariant. Define $\tau : \Gamma_1 \rightarrow \Gamma_2$ by $\psi_0(\mathfrak{g}_\alpha) = \mathfrak{g}_{\tau(\alpha)}$. It is a norm-preserving bijection. Thus $(\Gamma_1, \Gamma_2, \Gamma_3)$ is a generalized Belavin-Drinfeld triple. It is clear that Γ_3 is the largest subset of $\Gamma_1 \cap \Gamma_2$ stable under τ , and that $\psi_0 : \bar{\mathfrak{n}}_+(\Gamma_3) \rightarrow \bar{\mathfrak{n}}_+(\Gamma_3)$ is a Lie algebra isomorphism. Finally, it is easy to see that the map φ is obtained from this data by formulas

$$\begin{aligned} \varphi(\lambda)(e_\alpha) &= \frac{1}{2}e_\alpha & (\alpha \notin \langle \Gamma_1 \rangle) \\ \varphi(\lambda)(e_\alpha) &= \frac{1}{2}e_\alpha + \frac{\psi_0}{1 - e^{(\alpha, \lambda)}\psi_0}(e_\alpha) & (\alpha \in \langle \Gamma_1 \rangle) \end{aligned}$$

Conversely, it is clear how to construct from a generalized Belavin-Drinfeld triple $(\Gamma_1, \Gamma_2, \tau)$ the subalgebras $\mathfrak{n}_+(\Gamma_1)$, $\mathfrak{n}_+(\Gamma_2)$, $\mathfrak{n}_+(\Gamma_3)$ and, for each choice of Chevalley generators, a Lie algebra isomorphism $\psi_0 : \mathfrak{n}_+(\Gamma_1) \rightarrow \mathfrak{n}_+(\Gamma_2)$, and the map $\varphi(\lambda)$. Condition (16) on the $\mathfrak{h}_0 \otimes \mathfrak{h}_0$ -part comes from (21)-see [BD].

6 Examples

6.1 Constant r-matrices

Our results imply the following:

Corollary 1 *A dynamical r-matrix associated to a generalized Belavin-Drinfeld triple $(\Gamma_1, \Gamma_2, \tau)$ is gauge equivalent to a constant r-matrix if and only if $\Gamma_3 = \emptyset$.*

6.2 \mathfrak{h} -invariant dynamical r-matrices

When $\mathfrak{l} = \mathfrak{h}$, our classification coincides with that given in [EV]: the only \mathfrak{h} -graded generalized Belavin-Drinfeld triple is of the form $(\Gamma, \Gamma, \tau = Id)$. The

dynamical r-matrices obtained are then (up to gauge transformations and choice of Chevalley generators):

$$r(\lambda) = \frac{\Omega}{2} + \sum_{\alpha \in \Delta_+, \alpha \notin \langle \Gamma \rangle} \frac{1}{2} e_\alpha \wedge e_{-\alpha} + \sum_{\alpha \in \langle \Gamma \rangle \cap \Delta_+} \frac{1}{2} \coth\left(\frac{1}{2}(\alpha, \lambda)\right) e_\alpha \wedge e_{-\alpha}.$$

6.3 Example for \mathfrak{sl}_3 and \mathfrak{sl}_n

The first nontrivial example is for $\mathfrak{g} = \mathfrak{sl}_3$: fix a polarization $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\gamma \in \Delta} \mathfrak{g}_\gamma$ where $\Delta^+ = \{\alpha, \beta, \alpha + \beta\}$ and set $\mathfrak{l} = \mathbb{C}h_\rho$. Consider the generalized Belavin-Drinfeld triple with $\Gamma_1 = \Gamma_2 = \{\alpha, \beta\}$ and $\tau : \alpha \mapsto \beta, \beta \mapsto \alpha$. In this case, we can choose the map ψ_0 to be the following

$$\begin{aligned} e_\alpha &\mapsto e_\beta, & h_\alpha &\mapsto h_\beta, & e_{-\alpha} &\mapsto e_{-\beta} \\ e_\beta &\mapsto e_\alpha, & h_\beta &\mapsto h_\alpha, & e_{-\beta} &\mapsto e_{-\alpha} \\ e_{\alpha+\beta} &\mapsto -e_{\alpha+\beta}, & & & e_{-\alpha-\beta} &\mapsto -e_{-\alpha-\beta}. \end{aligned}$$

The corresponding dynamical r-matrix is given by:

$$\begin{aligned} r(\lambda) &= \frac{\Omega}{2} + r_{\mathfrak{h}_0, \mathfrak{h}_0} + \frac{1}{2} \coth(\alpha, \lambda) e_\alpha \wedge e_{-\alpha} + \frac{1}{2} \coth(\beta, \lambda) e_\beta \wedge e_{-\beta} \\ &\quad + \frac{1}{2} th(\alpha + \beta, \lambda) e_{\alpha+\beta} \wedge e_{-\alpha-\beta} + \frac{1}{2 \sinh(\alpha, \lambda)} e_\beta \wedge e_{-\alpha} \\ &\quad + \frac{1}{2 \sinh((\alpha, \lambda))} e_\alpha \wedge e_{-\beta}. \end{aligned} \quad (23)$$

This dynamical r-matrix is gauge-equivalent to the dynamical r-matrix

$$\begin{aligned} \tilde{r}(\lambda) &= \frac{\Omega}{2} + r_{\mathfrak{h}_0, \mathfrak{h}_0} + r_{\mathfrak{l}, \mathfrak{h}_0} - r_{\mathfrak{l}, \mathfrak{h}_0}^{21} + \frac{1}{2} \coth(\alpha, \lambda) e_\alpha \wedge e_{-\alpha} + \frac{1}{2} \coth(\beta, \lambda) e_\beta \wedge e_{-\beta} \\ &\quad + \frac{1}{2} th(\alpha + \beta, \lambda) e_{\alpha+\beta} \wedge e_{-\alpha-\beta} + \frac{e^{(\alpha, \lambda)}}{2 \sinh(\alpha, \lambda)} e_\beta \wedge e_{-\alpha} \\ &\quad + \frac{e^{-(\alpha, \lambda)}}{2 \sinh(\alpha, \lambda)} e_\alpha \wedge e_{-\beta}. \end{aligned} \quad (24)$$

when

$$(\alpha \otimes 1 + 1 \otimes \tau(\alpha))(r_{\mathfrak{h}_0, \mathfrak{h}_0} + r_{\mathfrak{l}, \mathfrak{h}_0} - r_{\mathfrak{l}, \mathfrak{h}_0}^{21}) = \frac{1}{2}(\alpha + \tau(\alpha))\Omega_{\mathfrak{h}}.$$

In particular, $\tilde{r}(\lambda)$ interpolates the constant r-matrix obtained from the Belavin-Drinfeld triple $(\Gamma_1 = \alpha, \Gamma_2 = \beta, \tau : \alpha \mapsto \beta)$ at $(\alpha, \lambda) \rightarrow \infty$ and the r-matrix obtained from $(\Gamma_1 = \beta, \Gamma_2 = \alpha, \tau : \beta \mapsto \alpha)$ at $(\alpha, \lambda) \rightarrow -\infty$.

Remark: The generalization of this example to $\mathfrak{g} = \mathfrak{sl}_{2n+1}$ is the following. Fix a polarization and let $\mathfrak{l} = \mathbb{C}h_\rho$. Denote by Δ the root system and by $\Pi = (\alpha_1, \dots, \alpha_{2n})$ the set of positive simple roots. Let $i : \alpha_k \mapsto \alpha_{2n+1-k}$ be the involution of the Dynkin diagram. The dynamical r-matrix obtained from the generalized Belavin-Drinfeld triple $(\Gamma_1 = \Gamma_2 = \Pi, \tau = i)$ interpolates the constant r-matrices obtained from the Belavin-Drinfeld triples $(\Gamma_1 = (\alpha_1, \dots, \alpha_n), \Gamma_2 = (\alpha_{n+1}, \dots, \alpha_{2n}), \tau = i)$ and $(\Gamma_1 = (\alpha_{n+1}, \dots, \alpha_{2n}), \Gamma_2 = (\alpha_1, \dots, \alpha_n), \tau = i^{-1})$.

6.4 Permutation dynamical r-matrices

Consider $\mathfrak{g} = \mathfrak{sl}_{2n}$, and let $\Pi = (\alpha_1, \dots, \alpha_{2n-1})$ denote a system of simple roots. For any $\sigma \in S_n$, we can construct a generalized Belavin-Drinfeld triple by setting $\Gamma_1 = \Gamma_2 = (\alpha_1, \alpha_3, \dots, \alpha_{2n-1})$ and $\tau : \alpha_{2k-1} \mapsto \alpha_{2\sigma(k)-1}$.

Acknowledgements: I heartily thank Pavel Etingof for his encouragements, constant help and for his communicative enthusiasm for mathematics. I also thank A. Varchenko and P. Etingof for sharing their work with me before publication, and Hung Yean Loke and Vadik Vologodsky for interesting discussions.

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